Problem 1. For each of the following logical expressions, state whether or not it is a tautology:

a) $((p \vee q) \wedge (\neg p \wedge \neg q)) \rightarrow q$ ☑ Tautology ☐ Not a tautology ☐ Don't know!

b) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ ☑ Tautology ☐ Not a tautology ☐ Don't know!

c) $(q \wedge (p \rightarrow q)) \rightarrow (p \wedge q)$ ☐ Tautology ☒ Not a tautology ☐ Don't know!

d) $((p \rightarrow q) \wedge (p \rightarrow r)) \rightarrow (p \rightarrow (q \wedge r))$ ☑ Tautology ☐ Not a tautology ☐ Don't know!

Problem 2. Let $P(x, y, z)$ denote the statement

$$x^2 + y^2 = z, \quad \text{where} \quad x, y, z \in \mathbb{Z}^+ .$$

What is the truth value of each of the following?

a) $\forall x \exists y \exists z P(x, y, z)$ ☑ True ☐ False ☐ Don't know!

$y = 1$
$z = x^2 + 1$

b) $\forall y \forall z \exists x P(x, y, z)$ ☐ True ☒ False ☐ Don't know!

$z = 3$
$y = 1$ $\Rightarrow x = \sqrt{2}$ which is not from $\mathbb{Z}^+$

c) $\forall x \forall y \exists z P(x, y, z)$ ☑ True ☐ False ☐ Don't know!

$z = x^2 + y^2$

d) $\forall z \exists x \exists y P(x, y, z)$ ☐ True ☒ False ☐ Don't know!

$z = 3$

Problem 3. For each of the arguments below, indicate whether it is valid or invalid.

a) All cheaters sit in the back row. George sits in the back row. Therefore George is a cheater.

☐ Valid ☒ Invalid ☐ Don't know!

b) For all students x, if x studies discrete math, then x is good at logic. Dawn is a student who studies discrete math. Therefore Dawn is good at logic.

☑ Valid ☐ Invalid ☐ Don't know!

c) If the compilation of a computer program produces error messages, then the program is not correct or the compiler is faulty. The compilation of this program does not produce error messages. Therefore this program is correct and the compiler is not faulty.

☐ Valid ☑ Invalid ☐ Don't know!

d) All students who do not do their homework and do not study the course material will not get a good course grade. John gets a good course grade. Therefore John did his homework or studied the course material.

☑ Valid ☐ Invalid ☐ Don't know!

Problem 4. (a) If the following is valid then give a proof, else give a counterexample:

For all positive $x \in \mathbb{R}$, if x is irrational and y is irrational then $x + y$ is irrational.

~~Prove it by contradiction.~~
~~Assume by contradiction that x, y are irrational and x+y is rational. I.e. $x+y = \frac{p}{q}$; $p, q \in \mathbb{Z}$~~

Counterexample: $-\pi, \pi$

(b) If the following statement is true then give a proof, else give a counterexample:

For all $k, m, n \in \mathbb{Z}^+$, if $k|mn$ then $k|m$ or $k|n$.

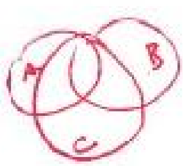
$$\left.\begin{array}{l} k = 6 \\ m = 2 \\ n = 3 \end{array}\right\} \text{Counterexample}$$

Problem 5. For each of the following, state whether or not the statement is True or False for general sets A, B, and C. ($\emptyset$ denotes the empty set.)

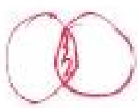
a) $(A \cup B) \cap (C \cup D) = (A \cap B) \cup (C \cap D)$ ☐ True ☒ False ☐ Don't know!

b) $A \cup (B - C) = (A \cup B) - (A \cup C)$ ☐ True ☒ False ☐ Don't know!

c) $(A \cap B) \subseteq C \Rightarrow (A - C) \cap (B - C) = \emptyset$ ☑ True ☐ False ☐ Don't know!



d) $(A \cup B) - (A \cap B) = \emptyset \Rightarrow A = B$ ☑ True ☐ False ☐ Don't know!



Problem 6.

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = x^3 - x$. Determine whether or not f is invertible:

☐ Invertible ☑ Not invertible ☐ Don't know!

It is not one-to-one

$f(0) = f(1) = f(-1) = 0$.

(b) Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be given by $f(m, n) = (m + n, n)$. Is f invertible?

☑ Invertible ☐ Not invertible ☐ Don't know!

(c) Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Is $f : S \longrightarrow S$ given by $f(k) = (8k+7) \bmod 10$ invertible?

☐ Invertible ☑ Not invertible ☐ Don't know!

? one-to-one ?

0 – 7
1 – 5
2 – 3
3 – 1

4 – 3
5 – 7

NOT ONE TO ONE !

d) If A and B are sets and $f : A \longrightarrow B$, then for any subset T of B its *pre-image* is defined as as $f^{-1}(T) = \{a \in A : f(a) \in T\}$, which is well-defined even when f does not have an inverse. Now let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = x^3$, and let $T = \{x \in \mathbb{R} : -8 < x \leq 1\}$. What is $f^{-1}(T)$? Enter your answer in the box below:

$\boxed{f^{-1}(T) = \{x \in \mathbb{R} : -2 < x \leq 1\}.}$

Problem 7. Define the predicates $L(x)$, $H(x)$, and $A(x)$ as follows:

$L(x) \equiv x$ *attends all Lectures*, $H(x) \equiv x$ *does all Homework*, $A(x) \equiv x$ *gets an A in the course.*

(a) In the space below write down the statement "*If Cindy gets an A in the course then she has attended all lectures or she has done all homework*" in logical form, using the predicates above, and using c to denote "*Cindy*" ; for example $A(c)$ denotes "Cindy gets an A in the course".

$$A(c) \rightarrow (L(c) \vee H(c))$$

(b) Write down the *contrapositive* of the statement in (a) in logical form, using the predicates defined above, as well as an equivalent sentence in English:

$$\neg(L(c) \vee H(c)) \rightarrow \neg A(c) \equiv (\neg L(c) \wedge \neg H(c)) \rightarrow \neg A(c)$$

If Cindy ~~does~~ did not attend all Lectures and she did not do all Homework then she does not get an A in the course.

(c) In the space below write the statement "*There is a student who did not attend all lectures but who did all homework, and who got an A in the course*" in logical notation, using quantifiers and the predicates defined above:

$$\exists x\,(\neg L(x) \wedge H(x) \wedge A(x))$$

(d) If P is a logical statement then its *negation* is the logical statement $\neg P$. Write down the *negation* of the statement in (c) in its simplest logical form, using quantifiers, and the predicates defined above. In particular, your negation must be written *in a form that does not start with* $\neg\exists$:

$$\forall x\, \neg(\neg L(x) \wedge H(x) \wedge A(x)) \equiv \underline{\forall x\, (L(x) \vee \neg H(x) \vee \neg A(x))}$$

Problem 8.

(a) When an integer n is divided by 7, the remainder is 5. What is the remainder when $9n$ is divided by 7 ? Enter your answer in the box:

$\boxed{3}$

$n = k \cdot 7 + 5$

$9n = 9k \cdot 7 + 9 \cdot 5$

$9n = 9k \cdot 7 + 45$

$9n = 9k \cdot 7 + 6 \cdot 7 + 3 = (9k+6)7 + 3.$

(b) Prove that there are no integer solutions x and y to the equation $x^2 - 5y = 2$.
Hint: What are the possible values of x^2 mod 5 ?

x^2 mod 5 ?

We know that $(x \cdot y) \bmod n = (x \bmod n) \cdot (y \bmod n) \bmod n.$

$x^2 \bmod 5 = (x \bmod 5)(x \bmod 5) \bmod 5$

Case I : $x \bmod 5 = 0 \Rightarrow x^2 \bmod 5 = 0$

Case II : $x \bmod 5 = 1 \Rightarrow x^2 \bmod 5 = 1$

Case III : $x \bmod 5 = 2 \Rightarrow x^2 \bmod 5 = 4$

Case IV : $x \bmod 5 = 3 \Rightarrow x^2 \bmod 5 = 3 \cdot 3 \bmod 5 = 4$

Case V : $x \bmod 5 = 4 \Rightarrow x^2 \bmod 5 = 4 \cdot 4 \bmod 5 = 1$

It means that there is no way how ~~the~~ we can get the remainder 2.

$x^2 = 5 \cdot y + 2 \; ; \; y \in \mathbb{Z}.$

Problem 9. DO ONLY ONE OF THE TWO PROBLEMS BELOW:

Put a circle around the one you choose to do: Choice (1) or Choice (2) .

Choice (1) The Fibonacci numbers are defined as $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$. Give a proof by induction to show that $\sum_{k=1}^{n} f_{2k-1} = f_{2n}$, for all $n \geq 1$. (Hint: This can be done by regular induction.)

Choice (2) Suppose $a_1, a_2, a_3, \ldots$ is a sequence defined as $a_1 = 1$, and $a_n = 2 \cdot a_{\lfloor n/2 \rfloor}$, when $n \geq 2$. Prove that $a_n \leq n$ for all integers $n \geq 1$ (Hint: This can be done by strong induction.)

$f_0 = 0$
$f_1 = 1$
$f_2 = 1$
$f_3 = 2$
$f_4 = 3$

(a) (1 point) Check that the base case is True:

1. $n=2$, $\sum_{k=1}^{2} f_{2k-1} = f_1 + f_3 = f_1 + f_1 + f_2 = f_1 + f_1 + f_1 + f_0 = 3 = f_4$ ✓

2. $n=2$: $a_2 = 2 \cdot a_{\lfloor 1 \rfloor} = 2 \cdot a_1 = 2 \leq 2$ ✓

(b) (1 point) Write down *very precisely* your inductive hypothesis, and what you will show in the inductive step (c) below.

1. IH: $\forall n \leq m$ $\sum_{k=1}^{n} f_{2k-1} = f_{2n}$ TO SHOW: $\sum_{k=1}^{n+1} f_{2k-1} = f_{2(n+1)}$

2. IH $\forall n \leq m$ $a_n \leq n$ TO SHOW: $a_{m+1} \leq m+1$

(c) (2 points) Carry out the actual proof of the inductive step:

1. $\sum_{k=1}^{n+1} f_{2k-1} = \sum_{k=1}^{n} f_{2k-1} + f_{2(n+1)-1} \overset{IH}{=} f_{2n} + f_{2n+2-1} = f_{2n} + f_{2n+1} = f_{2n+2}$ ✓

2. Case I: ~~a_{n+1}~~ $n+1$ – even. $a_{n+1} = 2 \cdot a_{\lfloor \frac{n+1}{2} \rfloor} = 2 \cdot a_{\frac{n+1}{2}} \overset{IH}{\leq} 2 \cdot \frac{n+1}{2} = n+1$ ✓ ($\frac{n+1}{2} < n$) So we can use our IH

Case II: $n+1$ – odd $a_{n+1} = 2 \cdot a_{\lfloor \frac{n+1}{2} \rfloor} = 2 \cdot a_{\frac{n}{2}} \overset{IH}{\leq} 2 \cdot \frac{n}{2} = n < n+1$ ✓

Problem 10.

(a) Determine whether the relation R on $\mathbb{Z}^+$, where xRy if and only if $x|(x+y)$, is *reflexive*:

☑ Reflexive ☐ Not reflexive ☐ Don't know!

$x|(x+x) = x|2x$ ✓

(b) Is the relation R on $\mathbb{Z}^+$, where xRy if and only if $x|(x+y)$, *symmetric* ?

☐ Symmetric ☑ Not symmetric ☐ Don't know!

$x|(x+y) \overset{?}{\Rightarrow} y|(x+y)$

No: $(2,6)$: $2|2+6$ but $6\nmid(2+6)$

$(2,6)\in R$ but $(6,2)\notin R$.

(c) Determine whether the relation R on $\mathbb{Z}^+$, where xRy if and only if $x|y$, is *antisymmetric*:

☐ Antisymmetric ☐ Not antisymmetric ☐ Don't know!

Def: $\forall xy\ (x,y)\in R \wedge x\neq y \Rightarrow (y,x)\notin R$.

If $x|y$ then $x\leq y$. Since $x\neq y$ then $x<y$. Then however $y\nmid x$ because a bigger number cannot divide a smaller one.

(d) Is the relation R on the set $\{a,b,c\}$ represented by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ *transitive* ?

☐ Transitive ☑ Not transitive ☐ Don't know!

Counterexample: $(1,2)\in R \wedge (2,1)\in R$ but $(1,1)\notin R$

Problem 11.

(a) The relation R on $\mathbb{R}$, defined as xRy if and only if $|x| - |y| = 0$ is:

☑ An equivalence relation ☐ Not an equivalence relation ☐ Don't know!

equivalence – reflexive $\wedge$ symmetric $\wedge$ transitive

reflexive ✓ $|x|-|x|=0$

symmetric $|x|-|y|=0 \Rightarrow |y|-|x|=0$ ✓

transitive if $|x|=|y| \wedge |y|=|z| \Rightarrow |x|=|z|$ ✓

(b) The relation R on $\mathbb{Z}$, where xRy if and only if $x \equiv y \pmod{13}$ is:

☑ An equivalence relation ☐ Not an equivalence relation ☐ Don't know!

reflexive: $x \equiv x \pmod{13}$ ✓ $x \equiv y \pmod{13}$ IFF $x \bmod 13 = y \bmod 13$

symmetric: $x \bmod 13 = y \bmod 13 \Rightarrow y \bmod 13 = x \bmod 13$ ✓

transitive: $x \bmod 13 = y \bmod 13 = z \bmod 13 \Rightarrow x \bmod 13 = z \bmod 13$.

(c) The relation R on $\mathbb{R}$, where xRy if and only if $x \leq y^2$ is:

☐ A partial order ☑ Not a partial order ☐ Don't know!

reflexive. $x \leq x^2$ ✓

anti-symmetric. $x \leq y^2 \wedge x \neq y \Rightarrow y \overset{?}{\leq} x^2$ NO: $(2,3) \in R \wedge 2 \neq 3 \wedge (3,2) \in R$

transitive.

(d) The relation R on $\mathbb{Z}^+$, where xRy if and only if $x|y$ is:

☑ A partial order ☐ Not a partial order ☐ Don't know!

reflexive. $x|x$ ✓

anti-symmetric. $x|y \wedge x \neq y \Rightarrow y \nmid x$ ✓

Yes. since if $x|y$ then $x \leq y$
Since $x \neq y$ then $x < y$. Bigger number cannot divide a smaller one.

transitive: if $x|y$ and $y|z$. then $x|z$ ✓

Problem 12.

Let R be the relation on $\mathbb{R}$ defined by xRy if and only if $xy = 1$. Thus R can also be represented as $\{(x, y) : xy = 1\}$. Use a similar representation for each of your answers to the questions below, and write your answer in the accompanying box.

(a) What is the composite relation R^2 ?

$\boxed{\{(x,z)\in\mathbb{R}^2 \mid x=z,\, x\neq 0\}.}$

$$R^2 = R\circ R \overset{DEF}{=} \{(x,z)\in\mathbb{R}^2 \mid \exists y\, (x,y)\in R \wedge (y,z)\in R\} =$$
$$= \{(x,z)\in\mathbb{R}^2 \mid \exists y ((xy=1)\wedge(yz=1))\} =$$
$$= \{(x,z)\in\mathbb{R}^2 \mid \exists y\, (x=z \wedge x\neq 0)\}$$
$$= \{(x,z)\in\mathbb{R}^2 \mid x=z \wedge x\neq 0\}.$$

$\Rightarrow$ $xy=1$, $yz=1$ } system of two equations

$y = \frac{1}{x}$ $(x\neq 0)$ $\Rightarrow$ $\frac{1}{x}z=1$ $(x\neq 0)$ $\Rightarrow$ $\boxed{x=z}$

$yz=1$

(b) What is the composite relation R^3 ?

$\boxed{1.\ R\ ;\ 2.\ \{(x,z)\in\mathbb{R}^2 \mid xz=1\} \text{ both are possible}}$

$$R^3 = R^2\circ R \overset{DEF}{=} \{(x,z)\in\mathbb{R}^2 \mid \exists y\, (x,y)\in R^2 \wedge (y,z)\in R\} =$$
$$= \{(x,z)\in\mathbb{R}^2 \mid \exists y\, (x=y, x\neq 0)\wedge yz=1\}$$
$$= \{(x,z)\in\mathbb{R} \mid \exists y\, (xz=1)\}$$
$$= \{(x,z)\in\mathbb{R} \mid \cancel{\exists y}\, xz=1\} = R$$

$x=y$
$yz=1$
$xz=1$ $\quad \underline{x\neq 0}$ I do not have to consider $x\neq 0$ since if $xz=1$ then $x\neq 0$ for sure

(c) What is the composite relation R^4 ?

$\boxed{R^2}$

$$R^4 = R^3\circ R = R\circ R = R^2$$

(d) What is the transitive closure of R ?

$\boxed{1.\ R\cup R^2 \quad 2.\ \{(x,z)\in\mathbb{R}^2 \mid xz=1 \vee (x=z\wedge x\neq 0)\}}$

Transitive closure is defined as

$$\bigcup_{n=1}^{\infty} R^n = R\cup R^2\cup R^3\cup R^4\cup R^5\cup \ldots$$
$$= R\cup R^2\cup R\cup R^2\cup R\cup \ldots$$
$$= R\cup R^2$$
$$= \{(x,z)\in\mathbb{R}^2 \mid xz=1 \vee (x=z \wedge x\neq 0)\}$$